of the actual velocity field $n_i(x_i)$. Furthermore the components n_i must be constant during the deformation process. This fact has not been clearly stated in the literature leading to some misinterpretations [for example, in Ref. 2, formula (30) is tensorially incorrect]. Usually n_i are known in problems in which $\dot{u}_i$ has only one component. This situation refers to all illustrative examples so far considered in the literature. This problem has been thoroughly discussed in the author's forthcoming paper.[3]

A Modified Proof

In Ref. 4, a simple lower bound theorem was proved assuming the following form of the time variable amplitude

$$\dot{T}(t) = \max_{x_i \in V} \|\dot{u}_i\| \tag{4}$$

This assumption has a direct physical motivation. First note that the general inequality (8) of Ref. 2 reduces to the equality if $\dot{u}_i = \dot{u}_i^*$. At that instant the left-hand side represents the energy dissipated in the course of plastic deformations while the right-hand side becomes an initial kinetic energy input. The above conclusion does not apply to formula (2) because an approximation, Eq. (1), has already been introduced. However, both sides of the inequality (2) can be brought as close as possible to each other by considering the amplitude $\dot{T}(t)$ to be identical with actual velocity field in the dynamic process at a certain point of the body [formula (4)]. The resulting bound would then be the most exact one.

With the assumption (4), the inequality (2) yields a simple lower bound theorem

$$\delta \geqq \tfrac{1}{2} V_o t_f^* \tag{5}$$

where $V_o = \|\dot{u}_i^o\|$. The preceding formula is valid in the case of uniformly distributed initial velocity. A corresponding result for an arbitrary distribution of initial velocity $\dot{u}_i^o(x_i)$ together with further details of the procedure, can be found in Ref. 4.

References

[1] Martin, J. B., "Impulsive Loading Theorem for Rigid-Plastic Continua," *Journal of the Engineering Mechanics Division*, Vol. 90, No. 5, Oct. 1964, pp. 27–42.

[2] Morales, W. J. and Nevill, G. E., "Lower Bounds on Deformations of Dynamically Loaded Rigid-Plastic Continua," *AIAA Journal*, Vol. 8, No. 11, Nov. 1970, pp. 2034–2046.

[3] Wierzbicki, T., "Lower Bound on Deformations in Impulsive Loading Problems," *Bulletin de l'Academie Polonaise de Sciences*, Vol. 19, No. 7–8, 1971, pp. 291–296.

[4] Wierzbicki, T., "Improved Lower Bound Theorem for Impulsively Loaded Continua," *Archives of Mechanics*, Vol. 23, No. 3, 1971, pp. 423–425.

Comment on "Vortices Induced in a Jet by a Subsonic Cross Flow"

A. M. Thompson*

Imperial College, London, England

A RECENT article by Durando[1] presented a small perturbation analysis of the flowfield produced some distance downstream of a jet issuing into a cross flow. This method, which is applicable only in regions where the plume makes a small angle to the mainstream, replaces the contra-rotating distributed vortices which are known to dominate the flow in this region by a pair of concentrated vortices connected by a vortex sheet. This analysis is closely akin to the very successful method used initially by Brown and Michael[2] for the flowfield about a slender delta wing with leading-edge separation and with it the author was able to predict the relationship between the vortex strength (Γ) and the distance along its path (ξ) in terms of similarity parameters, viz.

$$\Gamma^* = (0.79)/\chi^{1/3} \tag{14†}$$

where

$$\Gamma^* = \Gamma\sigma_e/4\pi U_\infty d_e \tag{13}$$

and

$$\chi = \xi\sigma_e/d_e \tag{7}$$

The empirical constant in Eq. (14) was obtained by comparison of the path and separation of the vortices with the experimental data of Pratte and Baines.[3] The model proved to be self consistent since matching the vortex separation to the jet spread led to the correct form for the path. However, as the author states, no comparison could be made between the predicted aand measured vortex strengths due to lack of data.

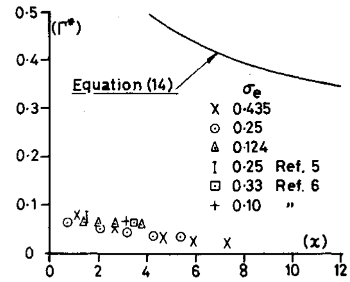

Fig. 1 Comparison between experimentally determined and semiempirical vortex strengths.

Comparison with Measured Circulations

An extensive experimental investigation into this problem has been made in the Department of Aeronautics, Imperial College,[4] and careful measurements of both the strength and path of the vortices have been made for a normal jet with velocity ratios (σ_e) of 0.435, 0.25, and 0.124. Calculations of the pressure field induced by these vortices on the wall from which the jet emerges show good quantitative agreement with the measured pressures. Considerable confidence is therefore felt in the accuracy of the vortex strength and path data.

A comparison between the experimental data and Eq. (14) is shown in Fig. (1). A result from Margason and Fearn[5] and two results from Tipping[6] are also indicated. The theoretical values of the vortex strength are an order of magnitude too high even for the region $\chi > 5$, which was quoted as the region for which the analysis might be expected to apply. The experimental values do not correlate on the basis of the similarity variables and there are no indications that there will be any improvement for higher χ values.

Comparison of the Lateral Separation of the Vortices

Because, again, of the lack of data, Durando was obliged to assume that the lateral spacing of the vortices varied in the same way as the jet cross section which had been measured by Pratte and Baines. This conflicts with the Imperial College data which correlates (Fig. 2) downstream of the immediate region of the nozzle in the form

$$(y_o/d_e)\sigma_e^{0.31} = f[(\xi/d_e)\sigma_e^{0.31}]$$

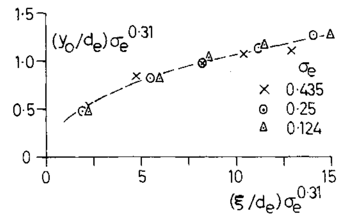

Fig. 2 Correlation of lateral vortex center position with velocity ratio.

Received June 14, 1971.

* Department of Aeronautics, Graduate of the Royal Aeronautical Society; presently Research Engineer, British Aircraft Corporation, Preton Division, Lancashire, England.

† Equation numbers refer to those appearing in Ref. 1.

thus drawing a much weaker dependance on the velocity ratio than that used by Durando in his Eq. (8). Hence, the basic form assumed for the spread of the vortices does not conform with the observed variations in vortex centre position and unfortunately, any modification to the form of Eq. (8) will be unrealistic since the self-consistency of the solution will be destroyed.

Conclusions

The proposed vortex model of Durando does not appear to be a satisfactory representation of the vortex dominated region of a deflected jet. In view of the success of the model when applied to separated flows on slender bodies this is disappointing. The failure is thought to be due to the very different vorticity distributions in the two flowfields. In the separated flows, well-defined vortex sheets feed the main vortices which develop with relatively compact cores. Thus, the mathematical analogy is closely representative of the real situation especially as the shape of the feeding vortex sheet does not have to be specified and can be curved as in the physical flow.

The distribution of vorticity in the jet plume is very different; the vortices being far more diffuse to the extent that the decay in circulation in the streamwise direction appears to be a direct consequence of the progressive overlapping of the two vorticity fields. This difference leads to the failure of Durando's theory on two counts. Firstly his model based on two concentrated cores is inadequate from the physical viewpoint of being unrepresentative of the real flow. Secondly, because of the cancellation of vorticity by overlapping, the circulation is far lower and changes in a manner quite different to that predicted by the criteria he uses. Unfortunately incorporating the observed behaviour of the vortices into his analysis destroys the self-consistency of the solution.

References

[1] Durando, N. A., "Vortices Induced in a Jet by a Subsonic Cross Flow," *AIAA Journal*, Vol. 9, No. 2, Feb. 1971, pp. 325–327.

[2] Brown, C. E. and Michael, W. H., "On Slender Delta Wings with Leading-Edge Separation," TN 3430, April 1955, NACA.

[3] Pratte, B. D. and Baines, W. D., "Profiles of the Round Turbulent Jet in a Crossflow," *Proceedings of the ASCE, Journal of the Hydraulics Division*, Nov. 1967, pp. 53–64.

[4] Thompson, A. M., "The Flow induced by Jets Exhausting Normally from a Plane Wall into an Airstream," Ph.D thesis, 1971, Univ. of London, England.

[5] Margason, R. J. and Fearn, R., "Jet Wake Characteristics and their Induced Aerodynamic Effects on V/STOL Aircraft in Transition Flight," NASA SP 218, Paper 1, Sept. 1969.

[6] Tipping, J. C., "Round Jets Issuing from a Flat Plate into a Cross Flow," Ph.D. thesis, 1965, Univ. of Bristol, England.

Reply by Author to A. M. Thompson

N. A. DURANDO*

McDonnell Aircraft Company, St. Louis, Mo.

IN his Comment, A. M. Thompson[1] shows large numerical discrepancies between his data and the predictions of the jet vortex model proposed by the author in Ref. 2. There is no question that Eq. (14) of Ref. 2 will greatly overpredict the strength of the counter-rotating vortices in the jet plume. A great deal of the discrepancy has been traced to errors made in adjusting the empirical constants of Ref. 2 to the jet plume characteristics given by Pratte and Baines,[3] and corrected values for these constant will be given below. Although satisfactory quantitative agreement is still not achieved, the author does not feel that Thompson's data as given in Ref. 1 may be used to conclusively prove or disprove the basic validity of his model. As stated in Ref. 2, the model is restricted to the "vortex zone" of the jet. As defined by Pratte and Baines,[3] this zone lies in the range

$$\chi > 5$$

where

$$\chi = \sigma_e \xi / d_e$$

Examination of Thompson's Figs. 1 and 2 reveals that only three of his data points fall in this range. In addition, it has been shown by Keffer and Baines[4] that for values of σ_e greater than approximately 0.25 (their parameter $R = 1/\sigma_e$). the jet plume is affected by the presence of the wall from which it emerges. Thompson's data at $\sigma_e = 0.435$ may therefore show the influence of this effect. As the range of validity of the model is approached, some of the trends exhibited by Thompson's data are not inconsistent with the conclusions of Ref. 2. In discussing these trends, conditions at fixed values of σ_e will be treated separately from the questions of correlations for different values of σ_e.

Received September 10, 1971.

* Senior Engineer, Aerodynamics.

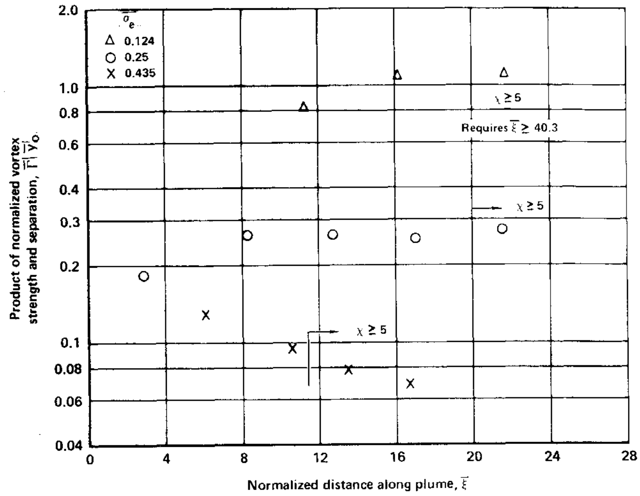

Fig. 1 Experimental values of the product of vortex strength and separation.

Conditions at a Fixed Value of σ_e

Equations (5) and (6) of Ref. 2 may be rewritten as follows:

$$\bar{\Gamma} = \bar{K}'/\bar{y}_o \qquad (1)$$

$$d\bar{z}_o/d\bar{\xi} = \bar{\Gamma}/y_o \qquad (2)$$

where

$$\bar{\xi} = \xi/d_e, \quad \bar{y} = y/d_e, \quad \bar{\Gamma} = \Gamma/4\pi U_\infty d_e, \quad \bar{K}'(\sigma_e) = K'/4\pi U_\infty d_e^2$$

Equation (1) is a direct result of the basic assumption that the net force on the vortices and their connecting sheet is zero. The product $\bar{\Gamma}\bar{y}_o$, calculated from Thompson's data,[1] is plotted in Fig. 1. Values of $\bar{\xi}$ corresponding to $\chi = 5$ are also indicated in the figure. For $\sigma_e = 0.124$, the greatest value of $\bar{\xi}$ calculated from Fig. 1 of Ref. 1, and that calculated from Fig. 2 of Ref. 1 disagreed by somewhat more than two jet diameters. Since this exceeded the discrepancies which might be expected from inaccuracies in reading the graphs, this point has not been included. Unfortunately, all values of $\bar{\xi}$ for $\sigma_e = 0.124$ lie well below that corresponding to the beginning of the vortex zone. For $\sigma_e = 0.25$, the product $\bar{\Gamma}\bar{y}_o$ is fairly constant, even for values of $\bar{\xi}$ below that corresponding to $\chi = 5$. This trend appears to support Eq. (1), and thus the basic assumption of Ref. 2.